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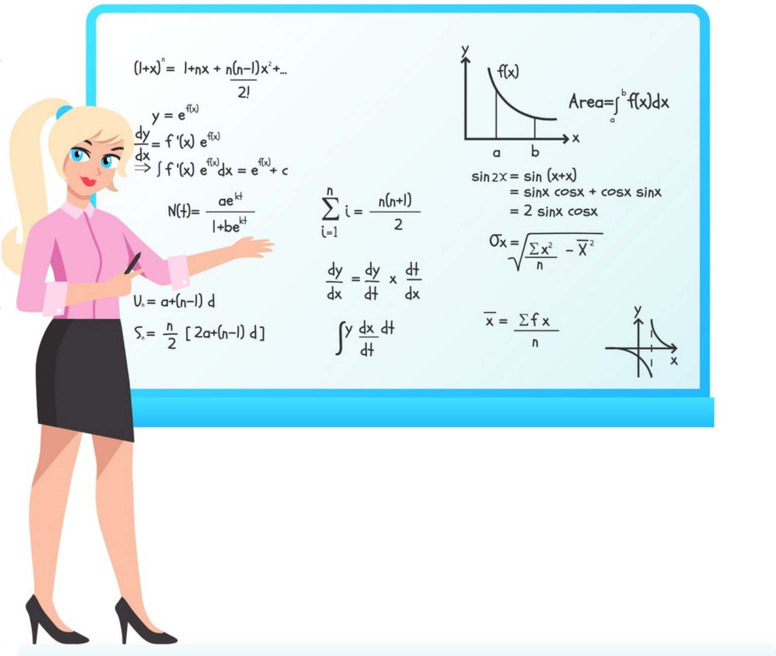
Complex Analysis

Introductory Notes



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Complex Analysis:

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When you study complex analysis and you realise that now you can trick your way through every real integration problem :
@aabillamematica
Oh, yeah. It's all coming together.


COMPLEX ANALYSIS
Tastes may have changed, but our secret recipe sure hasn't!
Triangle Inequality
Residue Theorem


A-Level students
Complex analysis
A-Level Foci

1 Topic Advice

Functions in complex analysis behave much better than functions in real analysis. Consider the definition of derivative at a point z_0, which is formally the same for real functions of a real variable and for complex functions of a complex variable:
$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$
In the real case, h is real, and approaches 0 along the real line, from the right and from the left. In the complex case, h is complex, and approaches 0 from any possible direction. This makes it more *difficult* for the limit to exist, and thus for complex functions of a complex variable to have a derivative. When such a function is viewed as a pair of real functions of two real variables, that is, when we write
$f(z) = u(x,y) + iv(x,y)$ (where u and v are real valued)
The existence of the above limit translates into the Cauchy-Riemann equations $\frac{\partial u}{dx} = \frac{\partial v}{dy}$ and $\frac{\partial u}{dy} = -\frac{\partial v}{dx}$
This again shows that a function of the form $u(x,y) + iv(x,y)$ is smooth in the complex sense only if it satisfies the rigidity imposed by the Cauchy-Riemann Equations.

In complex analysis, the notion of pointwise differentiability is not enough and we in fact need differentiability "in nearby neighbourhoods". A complex function $f(z)$ is said to be holomorphic at z_0 if there is a neighbourhood around z_0 in which f is differentiable. Every holomorphic function has a local representation as a convergent power series. While the definition "differentiable in a neighbourhood" is a suitably-equivalent way of saying "holomorphic" in the context of complex analysis, it is not necessarily the most informative way. The most informative definition of holomorphic is just the extension of the one from real analysis: a complex $f(z)$ is holomorphic at z_0 iff its Taylor series expansion at z_0 converges to $\mathrm{f(z)}$ in a neighbourhood of z_0. In complex analysis, this is equivalent to the statement that it is differentiable in a neighbourhood. In real analysis, it is not equivalent. In fact, in complex analysis the following equivalences hold - all (complex-)differentiable functions are (complex-)smooth and all smooth functions are holomorphic. However, the definitions of these terms need not be the same, even though we could do so because of these equivalences.

The crux of most complex analysis courses is **contour integration.** Get good at the techniques!
Methods include:

- Direct integration of a complex-valued function along a curve in the complex plane (a contour)
- Application of Cauchy Integral Formula
- Application of the Residue Theorem

This along with differentiability is explained in more detail in the Appendix.

2 Pre-Requisites

There aren't many pre-requisites except enough calculus to have covered partial differential equations, a course in real analysis and a little familiarity with metric spaces.

In detail:

- Sequences and series of numbers and of vectors
- Derivative in one variable
- Integration in one variable
- Integration with parameters
- Sequences and series of functions
- Uniform vs. pointwise convergence
- Derivative in several variables
- Line integrals
- Topology of metric spaces

Having a good basics of complex numbers mentioned below is vital!

2.1 Complex Numbers

2.1.1 Jargon

Consider $z = x + iy$, with $x, y \in \mathbb{R}$

- $x = Re(z)$ means the real part of z
- $y = Im(z)$ means the imaginary part of z
- z^* or $\bar{z} = x - iy$ is the complex conjugate of z

2.1.2 Important Results

- $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ (Note: $re^{i\theta}$ is Euler's form)

  Where:

  $|z| = r$

  $\arg z = \theta$

  $z^* = r(\cos\theta - i\sin\theta) = re^{-i\theta}$
- $z^n = r^n(\cos n\theta + i\ \sin n\theta) = r^n e^{in\theta}$ (by De Moivre's Theorem)
- $z^{-n} = r^n(\cos n\theta - i\ \sin n\theta) = r^n e^{-in\theta}$
- consider $z = \cos\theta + i\sin\theta = e^{i\theta}$

  $z + \frac{1}{z} = 2\cos\theta$

  $z - \frac{1}{z} = 2i\sin\theta$

  $z^n + \frac{1}{z^n} = z^n + z^{-n} = 2\cos n\theta$ so rearranging $\Rightarrow \cos n\theta = \frac{z^n + z^{-n}}{2}$

  $z^n - \frac{1}{z^n} = z^n - z^{-n} = 2i\sin n\theta$ so rearranging $\Rightarrow \sin n\theta = \frac{z^n - z^{-n}}{2i}$

2.1.3 Conjugate Rules

- $(z \pm w)^* = z^* \pm w^*$
- $(zw)^* = z^* w^*$
- $\left(\frac{z}{w}\right)^* = \frac{z^*}{w*}$ if $w \neq 0$

- $z.z^* = |z|^2$

2.1.4 Inequalities

- $|Re(z)| \leq |z|$ and $|Im(z)| \leq |z|$
- $|z + w| \leq |z| + |w|$
- $|z + w| \geq |z| - |w|$
- $|e^{real}| = e^{real}$

2.1.5 Modulus/Argument

- $|z_1 z_2| = |z_1||z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. This works like indices rules. When we multiply we add the powers

  So $[r_1(\cos\theta_1 + i\sin\theta_1)][r_2(\cos\theta_2 + i\sin\theta_2)]$ can be multiplied quickly and is

  $r_1 r_2[\cos(\theta_1 + \theta_2) \ + i\sin(\theta_1 + \theta_2) = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
- $arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ This works like indices rules. When we divide we subtract the powers

  So $\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$ can be divided quickly and is $\frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

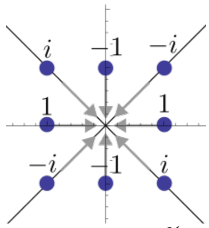
2.1.6 Loci

$|z - z_o| = r$ is a circle of radius r centred at z_0

3 Basics

3.1 Complex Differentiability

Differentiable in Complex Analysis does not simply mean differentiable from the left or the right like differentiation on the real line, we now look at all possible directions.



You will be familiar that differentiable at a point z_0 means $\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$, but now h is any tiny complex number and can go in ANY DIRECTION! Hence a derivative in two dimensions, rather than a derivative in one direction.

3.1.1 Differentiability Pointwise

$f(z) = u(x, y) + iv(x, y)$ (where u and v are real functions and $z = x + iy$) is **differentiable at z_0**, if

- the Cauchy Riemann Equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ hold at z_0 (this is a necessary, but not sufficient condition)
  Note: It is also necessary that u and v be real differentiable which is a stronger condition than the existence of the partial derivatives, but it is not necessary that these partial derivatives be continuous.

➢ How to show not pointwise complex differentiable?
Show the CRE's do not hold

$f(z) = 2x + ixy^2$
Determine whether the function is differentiable

$f(z) = u(x, y)$+$i\ v(x, y)$ so $u(x, y) = 2x$ and $v(x, y) = xy^2$

$\frac{\partial u}{\partial x} = 2$

$\frac{\partial u}{\partial y} = 0$

$\frac{\partial v}{\partial x} = y^2$

$\frac{\partial v}{\partial y} = 2xy$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2 = 2xy \Rightarrow xy = 1$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 0 = -y^2 \Rightarrow y = 0$

$f(z)$ is not differentiable (and thus obviously not holomorphic)

➢ How to show pointwise complex differentiable?
Show the CRE's hold
To find the points where complex differentiable, use the CRE's to set up equations and solve.

$f(z) = x^2 + iy^2$
Determine whether the function is differentiable

$f(z) = u(x,y) + i\,v(x,y)$ so $u(x,y) = x^2$ and $v(x,y) = y^2$

$\frac{\partial u}{\partial x} = 2x$

$\frac{\partial u}{\partial y} = 0$

$\frac{\partial v}{\partial x} = 0$

$\frac{\partial v}{\partial y} = 2y$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Longrightarrow 2x = 2y \Longrightarrow x = y$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Longrightarrow 0 = 0$

$f(z)$ is only differentiable at the points where $z = x + ix$

3.1.2 Differentiability In A Region

If $f(z) = u(x,y) + iv(x,y)$ (where u and v are real functions and $z = x + iy$) is differentiable **in a region R**, around z_0 iff and the following conditions are fulfilled in R

- the Cauchy Riemann Equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ hold at z_0
- $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, -\frac{\partial v}{\partial x}$ are continuous at z_0

Differentiable in a region R means holomorphic/analytic. See section 2.1.2.1 on holomorphic for more detail.

- How to show not holomorphic?
  Show the CRE's do not hold

- How to show holomorphic
  The Cauchy Riemann equations are not sufficient to show holomorphic, but they are necessary.
  To show the CRE's are **true for all values** in a region then we can say holomorphic i.e. if they hold for all (x,y) in that region. We also need the partial derivatives to be continuous.

$f(z) = z^2$
Determine whether the function is holomorphic.

$z = x + iy$
$f(z) = (x+iy)^2 = x^2 - y^2 + i2xy$
$f(z) = u(x,y) + i\,v(x,y)$ so $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$

$\frac{\partial u}{\partial x} = 2x$

$\frac{\partial u}{\partial y} = -2y$

$\frac{\partial v}{\partial x} = 2y$

$\frac{\partial v}{\partial y} = 2x$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and hold for all (x, y) and the partial derivatives are continuous so f is holomorphic

Note: In addition to CREs' you can also use the definition of derivative $\lim_{h\to 0} \frac{f(z_0+h)-f(h)}{h}$ and approach horizontally (real line) and vertically (imaginary line) and show these are the same (this is how we prove the CRE's).

Determine whether $\mathrm{f(z)} = |\mathrm{z}|^2$ is holomorphic

$z_0 = x_0 + iy_0$
$h = h_1 + ih_2$

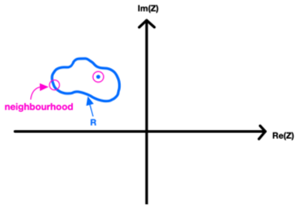
Horizontally: (real line)	Vertically: (imaginary line)
$\lim_{h_1\to 0} \frac{f(z_0+h_1)-f(z_0)}{z_0+h_1-z_0}$	$\lim_{h_2\to 0} \frac{f(z_0+ih_2)-f(z_0)}{z_0+ih_2-z_0}$
$= \lim_{h_1\to 0} \frac{f(z_0+h_1)-f(z_0)}{h_1}$	$= \lim_{h_2\to 0} \frac{f(z_0+ih_2)-f(z_0)}{ih_2}$
$= \lim_{h_1\to 0} \frac{\|z_0+h_1\|^2-\|z_0\|^2}{h_1}$	$= \lim_{h_2\to 0} \frac{\|z_0+ih_2\|^2-\|z_0\|^2}{ih_2}$
$= \lim_{h_1\to 0} \frac{\|x_0+iy_0+h_1\|^2-\|x_0+iy_0\|^2}{h_1}$	$= \lim_{h_2\to 0} \frac{\|x_0+iy_0+ih_2\|^2-\|x_0+iy_0\|^2}{ih_2}$
$= \lim_{h_1\to 0} \frac{({x_0}^2+h_1)^2+{y_0}^2-({x_0}^2+{y_0}^2)}{h_1}$	$= \lim_{h_2\to 0} \frac{{x_0}^2+(y_0+h_2)^2-({x_0}^2+{y_0}^2)}{ih_2}$
$= \lim_{h_1\to 0} \frac{{x_0}^2+2x_0h_1+{h_1}^2-{x_0}^2}{h_1}$	$= \lim_{h_2\to 0} \frac{{y_0}^2+2y_0h_2+{h_2}^2-{y_0}^2}{ih_2}$
$= \lim_{h_1\to 0} \frac{h_1(2x_0+h_1)}{h_1}$	$= \lim_{h_2\to 0} \frac{h_2(2y_0+h_2)}{ih_2}$
$= \lim_{h_1\to 0} 2x_0 + h_1$	$= \lim_{h_2\to 0} \frac{2y_0+h_2}{i}$
$= 2x_0$	$= -2y_0 i$

$2x_0 \neq -2y_0 i$ except at $(x_0, y_0) = (0,0)$

This is an example of complex variable function which is differentiable at a point (0,0) but not holomorphic at that point. This function has a complex derivative at $z = 0$, but nowhere else (to be holomorphic, it must differentiable in a neighbourhood of z_0). This is a great example to see that a function can be complex differentiable at just one point z_0 without being complex differentiable in an open set (neighbourhood) about that point).

3.1.2.1 Holomorphic

Holomorphic is defined with **neighbourhoods** etc and not fully equivalent to differentiability. The phrase "holomorphic at a point z" means not just differentiable at z, but differentiable everywhere within some open disk centred at z in the complex plane (in a neighbourhood of every point in a region R).



Note: A function can be differentiable without being holomorphic.

Something to think about: Many students tend to think that z^* is holomorphic, when it is in fact not. Try and prove why!

3.1.2.2 Analytic (aka regular)

Though the term "analytic function" is often used interchangeably with "holomorphic function", the word "analytic" is defined in a broader sense to denote any function (real, complex, or of more general type). Holomorphic functions are analytic and used interchangeably.
holomorphic↔analytic for complex analysis

If $f(z)$ is holomorphic/analytic in a region R, then the following holds in R
1) $f'(z), f''(z)$ …derivatives of all orders exist
2) $f(z)$ can be represented as a power series in a neighbourhood of each point in its domain with some non-zero radius of convergence that can be written as a convergent power series

3.1.2.3 Entire

Holomorphic implies differentiable on a certain given domain, whereas if a function is "entire", it is differentiable on **the whole complex plane** i.e. differentiable EVERYWHERE (ROC is infinite).

Be sure to understand the difference between differentiable and holomorphic and entire. Holomorphic means differentiable on a certain given domain, where entire means differentiable on ALL of $\mathbb{C}$.

3.2 Other Important Definitions

3.2.1 Harmonic

A function $u(x, y)$ is called harmonic if it is twice continuously differentiable and satisfies the following partial differential (Laplace) equation:
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
Note: If u and v satisfy the CRE's, then both u and v are harmonic, where $f(z) = u(x, y) + iv(x, y)$

3.2.2 Singularities, Poles and Zeros

A **zero** is just a place where the value of a complex function is zero. Formally we say, a point z_0 is called a zero of order n for the function $f(z)$ if $f(z)$ is analytic at z_0 and $f(z)$ and its first n − 1 derivatives vanish at z_0, but $f^{(n)}(z_0) \neq 0$ i.e. $f(z_0) = f'(z_0) = \cdots = f^{n-1}(z_0) = 0$ and $f^{(n)}(z_0) \neq 0$

In order to determine the order of a zero we just have to check how many times we need to derive a function until it is not equal to zero anymore.

$f(z) = (z + 4)^2$
Zero is $z = -4$
$f'(z) = 2(z + 4)$

$f'(-4) = 0$

$f''(-4) = 2 \neq 0$

So, order of zero here is 2

Notice how the **order of a zero is equal to the power a parenthesis is raised to**

A **singularity** is generally when a function is not defined in a point. There are three kinds of isolated singularities:

i. Removable singularities. These are functions which can be extended to a holomorphic function

f(z)= $\frac{z}{z}$
This is not defined at 0, but can be extended to a holomorphic function by letting f(0) = 1

ii. A **pole** is a point at which the function is undefined, as in infinite (this happens when you divide by 0, but the numerator is not 0).

$f(z) = \frac{3}{z-1}$ has a simple pole (pole of order 1) at $z = 1$

and

f(z) $= \frac{3}{(z-3)^2}$ has a double pole (pole of order 2) at $z = 3$

Formally we say, z_0 is a pole of order n of some analytic function $f(z)$ if $\lim_{z\to z_0}(z - z_0)^n \times f(z) \neq 0$ (if the power is less than n, the limit is "infinity" or does not exist. If the power is more than n, the limit is 0).

This is the same as saying we should be able to write $f(z)$ as $\frac{g(z)}{(z-z_0)^n}$ where $g(z_0) \neq 0$.

In other words, a pole z_0 is a value of z, so that $f(z_0) = \frac{g(z_0)}{0}$, hence $f(z) \to \infty$ as $z \to z_0$

Note: If z_0 is a zero of order n for $f(z)$, then z_0 is a pole of order n for $\frac{1}{f(z)}$

If $f(z)$ and $g(z)$ have zeros of orders m and n respectively at $z = z_0$, then $h(z) = f(z)g(z)$ has a zero of order $m + n$ at z_0.
If $f(z)$ and $g(z)$ have poles of orders m and n respectively at $z = z_0$, then $h(z) = f(z)g(z)$ has a pole of order $m + n$ at z_0.

Check power of parentheses to know order of pole

iii. **Essential singularities**. Essential singularities are singularities which are not removable nor poles. An example of this is:

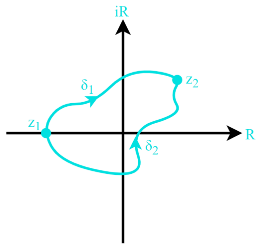
$$exp\left(\frac{1}{z}\right) = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots$$

3.3 Contour Integrals

3.3.1 What is a contour integral?

In complex analysis, contour integration is a method of evaluating certain integrals along paths in the complex plane. Why is the path chosen is important? Why do we need to know how we go from one point to another? In the good old days, when we only needed one axis to describe our input variables, the only way to go from one point to another was in a straight line.

Now in complex analysis, we need two axes to describe our input variables. This means that we are no longer limited to only one path and there are an infinite number of ways to do this since the only criteria we have is to go from the starting point to the endpoint. This is why it is important to know what kind of path you're taking from the starting point to the endpoint since different paths, in general, give different answers (unless the function has an antiderivative everywhere in the complex plane).



Formal definition
Let f : ℂ → ℂ be a continuous function on the directed smooth curve γ
Let z : ℝ → ℂ be any **parametrization** of γ that is consistent with its order (direction) i.e. if $z = z(t)$, $a \leq t \leq b$ is a parametrization of γ, then the line/path/contour integral along γ is denoted

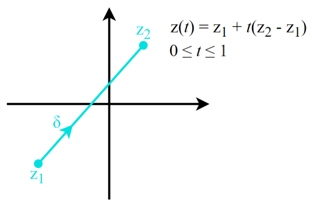
$$\int_{\gamma} \boldsymbol{f(z)dz} = \int_{a}^{b} [\boldsymbol{f}(\boldsymbol{z}(\boldsymbol{t})) \times \boldsymbol{z}'(\boldsymbol{t})]\, \boldsymbol{dt}$$

Make sure you understand that the product $f(z(t)) \times z'(t)$) is just a product of complex numbers.

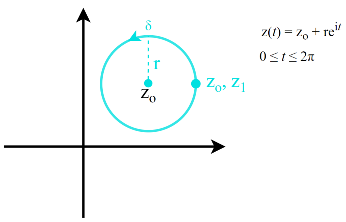
Note:

- A smooth curve is a curve that doesn't cross itself and has no sharp corners
- We need to find some parametrization $z(t)$ of γ, where a and b are the start and endpoints of the variable t.
  The standard parametrizations for three different smooth curves:

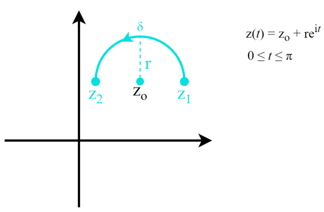
  Straight line



Circle



Semi-Circle



Notice how if you plug in the lower limit you get your start point and plugging in the upper limit you get your end point.

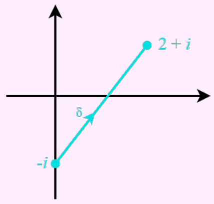
- Why is the path of γ so important? Why do we need to know how we go from one point to another? As mentioned previously, different paths give different answers unless the function has an antiderivative everywhere in the complex plane.

Common "Tricks":

- $\int_\gamma (f(z)+g(z))dz = \int_\gamma f(z)dz + \int_\gamma g(z)dz$
- $\int_\gamma f(z)dz = -\int_{-\gamma} f(z)dz$
- If $\gamma_3 = \gamma_1 + \gamma_2$, then $\int_{\gamma_3} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$
- Independence of paths:
  Some functions don't care which path you take when you integrate as long as the start and the endpoints are the same, integration along different paths will yield the same results.
  If the function is continuous on a contour γ which is made up of a number of directed smooth curves and the function has an antiderivative $F(z)$ on γ $(F'(z) = f(z))$, then $\int_\gamma f(z)dz = F(z_2) - F(z_1)$
  So, you can skip the parametrization since the only thing that matters is the starting point and the endpoint.

$\int_\gamma zdz$

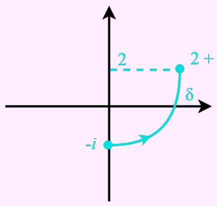
Straight line parametrization:



- $z(t) = z_1 + t(z_2 - z_1), \quad 0 \le t \le 1$
  $= -i + t(2 + i - -i)$
  $= -i + t(2 + 2i)$
- $z'(t) = 2 + 2i$

$$\begin{aligned} so \int_\gamma zdz &= \int_0^1 [-i + t(2+2i)][2+2i]dt \\ &= \int_0^1 [(2-2i) + (8i)t]dt \\ &= [(2-2i)t + 4it^2]_0^1 \\ &= 2 + 2i \end{aligned}$$

Quarter circle parametrization:



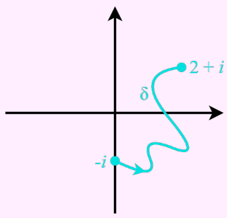
- $z(t) = i + 2e^{it} \quad \frac{3\pi}{2} \le t \le 0$
- $z'(t) = 2ie^{it}$

$$\begin{aligned} so \int_\gamma zdz &= \int_{\frac{3\pi}{2}}^0 (i + 2e^{it})(2ie^{it})dt \\ &= \int_{\frac{3\pi}{2}}^0 (-2e^{it} + 4ie^{2it})dt \end{aligned}$$

$= \left[2ie^{it} + 2e^{2it}\right]_{\frac{3\pi}{2}}^{0}$

$= 2 + 2i$

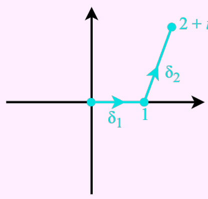
No parametrization:



No idea how to parametrize this $\int_\gamma z dz = \left[\frac{z^2}{2}\right]_{-i}^{2+i} = 2 + 2i$

Notice how all answers are the same. This function doesn't care which path you take as long as the start and endpoints are the same. This is because this function $f(z) = z$ has an antiderivative everywhere in the complex plane, so the only thing that really matters when doing the calculations are the start and endpoints.

$\int_\gamma \bar{z} dz$



γ_1:

- $z(t) = 0 + t(1-0) = t, \ \ 0 \le t \le 1$
- $z'(t) = 1$

γ_2:

- $z(t) = 1 + t(2 + i - 1), \ 0 \le t \le 1$
  $= 1 + t(1+i)$
- $z'(t) = 1 + i$

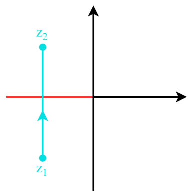
$$so \int_{\gamma_3 = \gamma_1 + \gamma_2} \bar{z} dz = \int_{\gamma_1} \bar{z} dz + \int_{\gamma_2} \bar{z} dz$$
$$= \int_0^1 t(1) dt + \int_0^1 [1 + t(1 - i)](1 + i) dt$$
$$= \frac{3}{2}$$

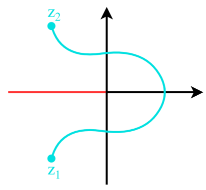
Note: No conjugate here, only $f(\bar{z})$ gets the conjugate

Note:

$$\int_\gamma \frac{1}{z} dz = \begin{cases} \int_a^b \frac{1}{z(t)} z'(t) dt & (1) \\ [\ln(z)]_{z_1}^{z_2} & (2) \end{cases}$$

(1) (2)





You may find it helpful to memorise the following: (if you have a closed contour)
$\int z^n dz = 0$ if $n \neq -1$

$\int z^n dz = \begin{cases} 2\pi i & (if\ you\ have\ a\ close\ contour\ which\ ocntains\ origin) \\ 0 & (if\ you\ have\ a\ close\ contour\ which\ doesn't\ contain\ origin) \end{cases}$ if $n = -1$

3.3.2 Cauchy's Theorem (aka Cauchy-Goursat)

If $f(z)$ is holomorphic everywhere within some simply connected region then $\int_c f(z)dz = 0$ for every simple closed path C lying in the region. We need $f(z)$ to be continuous and have no singularities in the region to apply this theorem!

This is perhaps the most important theorem in complex analysis!!!

3.3.3 Cauchy Integral Formula

Suppose C is a simple closed curve and the function f(z) is holomorphic on a region containing C and its interior. We assume C is oriented counter-clockwise. Then for any point z_0 **inside** C, $f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}dz$ where the integral is a contour integral along the contour γ enclosing the point z_0.
Note: z_0 can be the singularity point
So, $\int_\gamma \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$

If we remove the singularity z_0 then $\int_\gamma \frac{f(z)}{z-z_0}dz = 0$.
So, the main difference with Cauchy's Theorem and Cauchy's Integral formula is that we have to use Cauchy's Integral formula when dealing with a singularity.

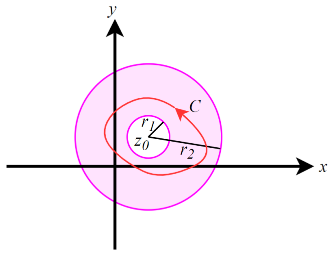
Cauchy's Integral Formula can be extended to Cauchy Integral formula for derivatives:
$f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}dz$
So, $\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(z)$

3.3.4 Laurent Expansion

One of the shortcomings of the Taylor series is that the circle of convergence is often only a part of the region in which $f(z)$ is analytic. As an example, the series
$1 + z + z^2 + z^3 + \ldots$ converges to $f(z) = \frac{1}{1-z}$ only inside the circle |z| = 1 even though f(z) is analytic everywhere except at z = 1.

The Laurent series is an attempt to represent $f(z)$ as a series over as large a region as possible. We expand the series around a point of singularity up to, but not including the singularity itself. In other words, the Laurent series still works if z_0 is an isolated singularity.

The figure below shows an annulus of convergence $r_1 < |z - z_0| < r_2$ within which the Laurent series (which is an extension of the Taylor series) will converge. The extension includes negative powers of $(z - z_0)$.



You will use this a lot for the Residue Theorem.

Find the Laurent expansion of $g(z) = \frac{z}{(1+z)(1-z)}$ in the following domains

i. $|z| < 1$
(hint: want to expand about z i.e. want in powers of z)

ii. $|z-1| > 2$
(hint: want in powers of z-1)

i. $\frac{z}{(1+z)(1-z)}$

$= \frac{-\frac{1}{2}}{1+z} + \frac{\frac{1}{2}}{1-z}$

$= -\frac{1}{2}\left(\frac{1}{1+z}\right) + \frac{1}{2}\left(\frac{1}{1-z}\right)$

$= -\frac{1}{2}(1 - z + z^2 - z^3 + z^4 + \cdots) + \frac{1}{2}(1 + z + z^2 + z^3 + z^4 + \cdots)$

$= z + z^3 + z^5 + \cdots$

ii. $\frac{z}{(1+z)(1-z)}$

$= \frac{-\frac{1}{2}}{1+z} + \frac{\frac{1}{2}}{1-z}$

$= \frac{-\frac{1}{2}}{z-1+2} + \frac{-\frac{1}{2}}{z-1}$

$= -\frac{1}{2}\left(\frac{1}{z-1+2}\right) - \frac{1}{2}\left(\frac{1}{z-1}\right)$ (*)

$\left(\frac{1}{z-1+2}\right)$ is not in the correct form that we need. However, $\left(\frac{1}{z-1}\right)$ is

Consider $\left(\frac{1}{z-1+2}\right)$:

$\left(\frac{1}{z-1+2}\right) = \left(\frac{\frac{1}{2}}{1+\frac{z-1}{2}}\right)$ if dividing by 2 and $\left(\frac{1}{z-1+2}\right) = \left(\frac{\frac{1}{z-1}}{1+\frac{2}{z-1}}\right)$ if dividing by $z-1$

Here we have $|z-1| > 2$ so $\frac{2}{|z-1|} < 1$ so use form 2

So * becomes

$= -\frac{1}{2}\left(\frac{\frac{1}{z-1}}{1+\frac{2}{z-1}}\right) - \frac{1}{2}\left(\frac{1}{z-1}\right)$

$$= -\frac{1}{2} \times \frac{1}{z-1}\left(\frac{1}{1+\frac{2}{z-1}}\right) - \frac{1}{2}\left(\frac{1}{z-1}\right)$$

$$= -\frac{1}{2(z-1)}\left(\frac{1}{1+\frac{2}{z-1}}\right) - \frac{1}{2}\left(\frac{1}{z-1}\right)$$

$$= -\frac{1}{2(z-1)}\left[1 - \frac{2}{z-1} + \left(\frac{2}{z-1}\right)^2 - \left(\frac{2}{z-1}\right)^3 + \cdots + \frac{(-1)^n 2^{n-1}}{(z-1)^n} + \cdots\right] - \frac{1}{2}\left(\frac{1}{z-1}\right)$$

$$= -\frac{1}{2(z-1)} \times \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z-1}\right)^n - \frac{1}{2}\left(\frac{1}{z-1}\right)$$

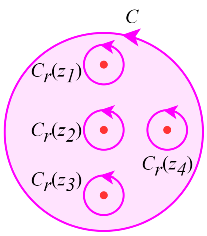
3.3.5 Residue Theorem

3.3.5.1 Definition

The residue theorem, sometimes called Cauchy's residue theorem, is a powerful tool to evaluate line integrals of analytic functions over closed curves; it can often be used to compute real integrals and infinite series as well. It generalizes Cauchy's Theorem and Cauchy's integral formula.

The numerator of the term $\frac{a_{-1}}{z-z_0}$, a_{-1} is called the residue of f at the pole z_0. We write $res(f, z_0) = a_{-1}$

What is a residue?
As already mentioned, the holomorphic functions have an extraordinary property: if you compute an integral along a path, the value of the integral does not depend on the path! More precisely, if the function is holomorphic everywhere inside a closed path, the integral is just zero. But if the function has poles (zeroes at the denominator, terms in the Laurent series), every pole brings a nonzero contribution called its residue. You can shrink the path as much as you want, even turning it to infinitesimal circles around every pole, provided you keep the poles in. So, the residues are what is left (as regards integration) after you removed all the holomorphic parts of the domain.



What's so special about the a^{-1} term in the Laurent expansion?" The answer can be stated very simply and can be understood without needing complex numbers: Every other term in the Laurent series integrates to a power function. That one doesn't.

3.3.5.2 Residue Theorem

Let f be holomorphic on an open set containing a contour γ and its interior except for the poles $z_1, z_2, \ldots z_k$ inside γ then $\int_\gamma f(z)dz = 2\pi i \sum residues$. This is called the residue formula.

3.3.5.3 Calculating Residues

There are 3 common techniques to calculate residues

1) Use limits and find residue at SIMPLE pole
$\frac{1}{(n-1)!} \lim_{z \to pole} \left[\frac{d^{n-1}}{dz^{n-1}}((z - pole)^n \times f(z))\right]$

Note: Instead of taking the limit you can write what is in the bracket as $\frac{\dots}{(z-pole)^n \dots}$ and evaluate … at f(singularity) i.e cover up pole in function and evaluate)

2) Expand about the pole (calculate derivatives)
Hint: Get the function in the form $\frac{\alpha}{(z-pole)^n}\left[\frac{g^{(n-1)}(pole)}{(n-1)!}\right]$

We only use this way if calculating derivatives is easy

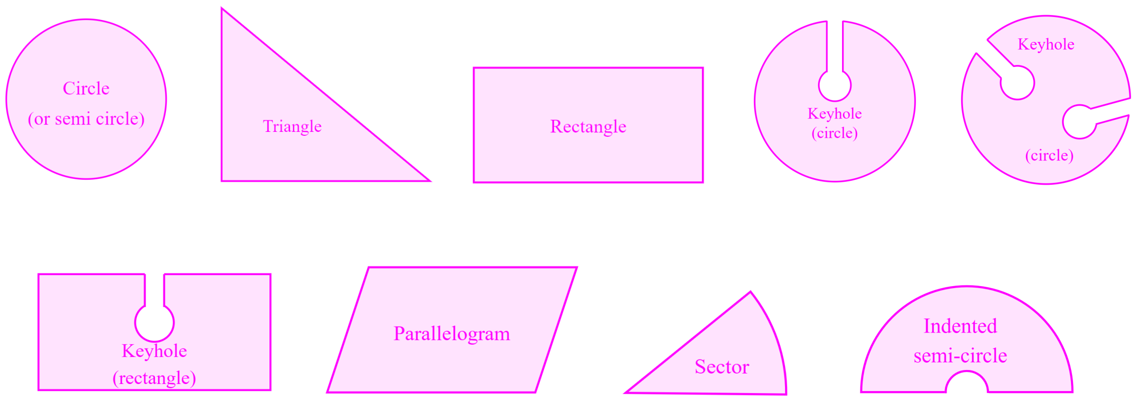
3) Expanding about the pole (use Taylor expansion or Laurent series). We usually use this when the power of the pole is high and hence calculating derivatives would be painful
$\frac{\alpha}{(z-pole)^n}\left[\begin{matrix} Pick\ out\ coefficient\ from \\ term\ with\ n-1\ power \end{matrix}\right]$
We usually get what is inside this bracket from putting in form $\frac{1}{1\pm z}$ form and expanding OR Laurent series

3.3.6 Techniques Summary

3.3.6.1 How To Choose a Contour and Contour Integration Techniques

- Remember the definitions
  A contour is a piecewise smooth curve
  A contour is called simple if it does not cross itself
  A contour c is closed if and only if the start point of c is equal to the end point of c
  A contour is defined as "made up of a finite number of smooth paths

- Common (toy) contours include



The four most commonly used are circle/semi-circle, triangle, indented semi-circle and keyhole. A rectangular contour depends on the nature of pole and residues of your function.

Quick hints:
- If $\int_0^{2\pi} trig$, use circular contour
- If $\int log$, use an indented semi-circle around origin
- If $\int complex\ number^{fractional\ power}$, use keyhole contour when we have branch cut
- If $\int_0^{\infty}$ with trig in it, write as $\int_{-\infty}^{\infty}$ and use
  - indented semi-circle if poles (with estimation lemma for some of the paths)
  - semi-circle if no poles

  Remember if $cosx$ we write as $Re(e^{ix})$ and if $sinx$ we write as $Im(e^{ix})$
- If $\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}$ or $\int_{-\infty}^{\infty}\frac{p(x)}{q(x)}e^{iax}$ i.e. have cos or sin in them and use
  - indented semi-circle if poles (with Estimation or Jordan's Lemma for some of the paths)
  - semi-circle if no poles

  $q(x)$ has degree 1 more than $p(x)$, use Jordan's Lemma to show top arc goes to 0

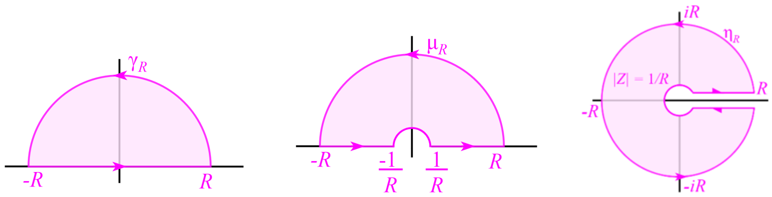
$q(x)$ has degree much bigger than $p(x)$, use estimation lemma to show top arc goes to 0

Get good at picking your own contour (remember to take contour anticlockwise). For a simple contour, the closed contour must enclose one or more poles otherwise the integral is zero.

Possible methods:

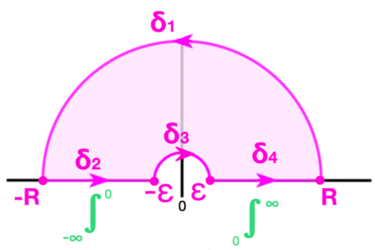
1) **pick a path and use $\int_\gamma f(z)dz = \int_a^b [f(z(t)) \times z'(t)]\, dt$**
2) **Pick a contour to avoid poles and use Cauchy's Theorem to say equals zero**
3) **Pick a contour with poles and use Cauchy Integral or Residue Theorem**
   Remember: sometimes we use the fact that we know the answer to work backwards from one of the paths giving what the integrand you're trying to find (see * below)

$\int_0^\infty$ or $\int_{-\infty}^\infty$ integrals:
One very common use for contour integrals is the evaluation of integrals along the real line that are not readily found by using only real variable methods. When choosing a contour to evaluate an integral on the real line, a contour is generally chosen based on the range of integration and the position of poles in the complex plane. Many cases can be solved by integrating around the top half of a circle with radius of infinity and integrating along the entire real line.



For example, for an integral from $-\infty$ to ∞along the real axis, the contour at left could be chosen if the function f had no poles on the real line, and the middle contour could be chosen if it had a pole at the origin. To perform an integral over the positive real axis from 0 to ∞ for a function with a pole at 0, the contour at right could be chosen.

Sometimes you can use the fact that you know the answer is zero from the Cauchy Theorem or whatever numerical answer is from Residue or Cauchy Integral Formula to work backwards to get the integral that you are asked to find which is from ONE of the paths or two of them combined.
*In the common sketch example below, we know $\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_3} + \int_{\gamma_4} = 0$ and we can work out $\int_{\gamma_1}$ and $\int_{\gamma_3}$ and cleverly combine $\int_{\gamma_2}$ and $\int_{\gamma_4}$ to give us what we want.
We do this for trig types $\int_0^\infty$ ... $= \frac{1}{2}\int_{-\infty}^\infty$ ... (where ... is trig)



γ_1: use Estimation Lemma or Jordan's Lemma to show $\to 0$
γ_4: Parametrize $z = t$
$\int_\varepsilon^R$ ...
γ_2: Parametrise $z = t$ to get $\int_{-R}^{-\varepsilon}$ ...
Substitute $z = -u$ to get $\int_\varepsilon^R$ ...
Put u back as t
Then **combine** with γ_4 to get
$2\int_\varepsilon^R$ ...
If done correctly ... should be same as ... given in question
Send $\varepsilon \to 0$, R$\to \infty$

$2\int_0^\infty \ \ldots$

γ_3: Either use parametrisation $z = \varepsilon e^{it}, 0 \le t \le \pi$ OR expansion of e^{iz} and then parametrization for one part and estimation for other part to show $\to -\pi$ or $-\pi i$

As mentioned, get good at the techniques for this course for contour integration as this is the crux of Complex analysis!! Methods include:

- Direct integration of a complex-valued function along a curve in the complex plane (a contour)
- Application of Cauchy Integral Formula
- Application of the Residue Theorem

The following topics will be stated only for further reading

3.4 Triangle Inequality

Suppose f(t) is a complex valued function of a real variable, defined on a ≤ t ≤ b. Then

- Triangle Inequality 1 for integrals
  $\left|\int_a^b f(t)dt\right| \le \int_a^b |f(t)|dt$
  This is an IMPORTANT Lemma and used to prove the estimation lemma

- Triangle Inequality 2 for integrals
  For any function f(z) and any curve γ, we have $\left|\int_\gamma f(z)dz\right| \le \int_\gamma |f(z)||dz|$
  This is also used to prove the estimation lemma below

3.5 Estimation Lemma (aka ML Inequality Theorem)

This is a very simple, yet very useful result in complex analysis
Let $f(z)$ be continuous on a curve γ. If $|f(z(t))| \le m$ i.e. m={$\max_{z\in\gamma}$ {f(z)} then
$\left|\int_\gamma f(z)dz)\right| \le m\, L(\gamma)$, where $L(\gamma)$ is the length of the curve γ
The proof of this uses the two inequalities in the section above

Estimation lemma and triangle inequality 1 are often used to show contour parts of an integral go to 0

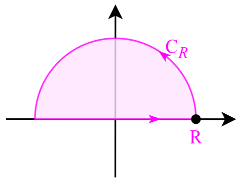
Some useful modulus results you will need when applying Estimation lemma are:
$|\mathrm{e}^{\mathrm{ia}}| = 1$ where a is real $\Rightarrow |\mathrm{re}^{\mathrm{ia}}| = |\mathrm{r}||\mathrm{e}^{\mathrm{ia}}| = |\mathrm{r}|$
$|\mathrm{e}^{\mathrm{iz}}| = |\mathrm{e}^{\mathrm{i(a+ib)}}| = |\mathrm{e}^{\mathrm{ia}}||\mathrm{e}^{\mathrm{i^2b}}| = |\mathrm{e}^{\mathrm{ia}}||\mathrm{e}^{-\mathrm{b}}| = |\mathrm{e}^{-\mathrm{b}}| = \mathrm{e}^{-\mathrm{im(z)}}$
so similarly, $|\mathrm{e}^{\mathrm{iRe^{i\theta}}}| = |\mathrm{e}^{\mathrm{i(Rcos\,\theta+iRsin\,\theta)}}| = |\mathrm{e}^{\mathrm{iRcos\theta}}||\mathrm{e}^{-\mathrm{Rsin}\,\theta}| = |\mathrm{e}^{-\mathrm{Rsin}\,\theta}|(1) = \mathrm{e}^{-\mathrm{Rsin\theta}}$
Note: $|\mathrm{e}^{\mathrm{iz}}| \le 1$ for $\mathrm{im(z)} \ge 0$ (if $\mathrm{im(z)} < 0$ then unbounded)

3.6 Cauchy Inequality

$|f^{(n)}(z_0)| \le \frac{n! \times \max_{|z-z_0|=R}(f(z))}{R^n}$
This comes from Cauchy Integral formula for n derivatives and then estimation lemma

3.7 Jordan's Lemma

This comes from Cauchy Integral formula for n derivatives and then estimation lemma
$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}dz = 0,$ where a is real
This says, consider arc getting bigger and bigger, if we integrate along the arc we end up with 0
Consider semi-circle or contour to integrate across.
Let's say upper arc (semi-circle arc with radius R).



If we integrate along arc c_R and take the limit as R approaches infinity the value of the arc does not contribute anything to our contour.

Note: Most commonly look at $f(z) = \frac{P(z)}{Q(z)}$ where $\deg Q - \deg P \geq 1$